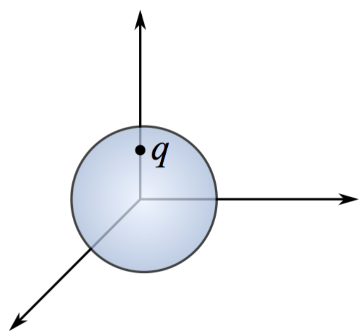
Practice Final Exam — I

The final will be a 180 minute open book, open notes exam. Do all four problems.

1. A point charge q is located on the z axis at $z = +d$. If this charge was isolated in free space, the electric potential could be expanded as

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0} \sum_{l\geq 0} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$$

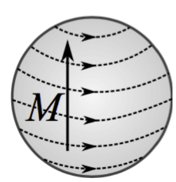
where $r_< = \min(r,d)$ and $r_> = \max(r,d)$. For this problem, however, the charge is located *inside* a dielectric sphere of permittivity ϵ and radius a (with $d < a$).



   (a) Find the electric potential everywhere as an expansion in Legendre polynomials.

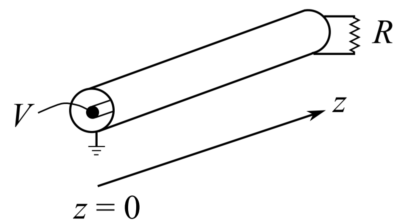
   (b) Show that, in the limit $\epsilon/\epsilon_0 \to \infty$ the electric potential outside the sphere reduces to that of a charged conducting sphere.

2. A wire coil is wound around the surface of a uniformly magnetized sphere of radius a. The permanent magnetization is given by $\vec{M} = M_0\hat{z}$, while the coil is designed so that it carries a surface current $\vec{K} = \hat{\phi}(I/a)\sin\theta$.



Find the magnetic induction $\vec{B}$ and magnetic field $\vec{H}$ everywhere. (You may want to introduce separate magnetic scalar potentials for the inside and the outside of the sphere.)

3. A long coaxial cable consists of an inner conductor of radius a surrounded by an outer conductor of radius b. A dielectric with permittivity ϵ and permeability μ fills the volume between the conductors. The outer conductor is held at zero potential, while a potential $V(t)$ is applied to the inner conductor at the $z = 0$ end of the cable. The far end of the cable is attached to a resistor. As a result, a current $I(t)$ flows along the inner conductor, through the resistor, and back along the outer conductor.



(a) If the potential $V(t) = V_0$ is constant, show that the electric and magnetic fields inside the cable are given in cylindrical coordinates (ρ, ϕ, z) by

$$\vec{E} = \hat{\rho}\frac{V_0}{\rho \log(b/a)}, \qquad \vec{H} = \hat{\phi}\frac{I_0}{2\pi\rho}$$

where $I_0 = V_0/R$. Ignore fringe effects.

(b) Now suppose the potential $V(t) = V_0 e^{-i\omega t}$ is harmonic but slowly varying in time. Due to symmetry considerations, the electric and magnetic fields will still point in the $\hat{\rho}$ and $\hat{\phi}$ directions, respectively. However, they may now pick up an additional dependence on the distance z along the cable. Find the electric and magnetic fields up to first order in the angular frequency ω.

(c) Find the power transmitted along the cable by integrating the time-averaged Poynting vector over a plane transverse to the cable (ie at a fixed value of z). Work up to first order in ω. How does this compare with the power dissipated in the resistor?

4. An inhomogeneous plane wave propagating in vacuum has an electric field given by the real part of

$$\vec{E}(\vec{x}, t) = \vec{\mathcal{E}} e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

where $\vec{\mathcal{E}}$ and $\vec{k}$ are complex and ω is real.

(a) Show that a circularly polarized inhomogeneous plane wave given by

$$\vec{k} = k(0, i\sinh\alpha, \cosh\alpha), \qquad \vec{\mathcal{E}} = E_0(\operatorname{sech}\alpha, i, \tanh\alpha), \qquad \omega = ck$$

satisfies all four Maxwell's equations. Here E_0, k and α are real constants.

(b) Compute the time-averaged Poynting vector $\vec{S}$ and time-averaged energy density u for this inhomogeneous plane wave, and find the velocity of the energy flow $\vec{v} = \vec{S}/u$.

(c) Show that the energy flow is not strictly along the z direction, but that the z component of the energy flow velocity matches the phase velocity of the wave.